SEEMOUS 2020 SOLUTIONS

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Problem 1. Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

$$(1) \qquad \begin{aligned} A + A^{\times} &= I_{2020}, \\ A \cdot A^{\times} &= I_{2020}, \end{aligned}$$

where $A^{\times}$ is the adjugate matrix of A, i.e., the matrix whose elements are $a_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A, eliminating the line j and the column i.

Find the maximum number of matrices verifying (1) such that any two of them are not similar.

Solution. It is known that

$$A \cdot A^{\times} = \det A \cdot I_{2020},$$

hence, from the second relation we get $\det A = 1$, so A is invertible. Next, multiplying in the first relation by A, we get

$$A^2 - A + I_{2020} = \mathcal{O}_{2020}.$$

It follows that the minimal polynomial of A divides

$$X^2 - X + 1 = (X - \omega)(X - \bar{\omega}),$$

where

$$\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.$$

Because the factors of the minimal polynomial of A are of degree 1, it follows that A is diagonalizable, so A is similar to a matrix of the form

$$A_k = \begin{pmatrix} \omega I_k & \mathcal{O}_{k,2020-k} \\ \mathcal{O}_{2020-k,k} & \bar{\omega} I_{n-k} \end{pmatrix}, \quad k \in \{0, 1, ..., 2020\}.$$

But $\det A = 1$, so we must have

$$\begin{aligned} \omega^k \bar{\omega}^{2020-k} &= 1 \Leftrightarrow \omega^{2k-2020} = 1 \Leftrightarrow \cos\frac{(2k-2020)\pi}{3} + i\sin\frac{(2k-2020)\pi}{3} = 1 \\ &\Leftrightarrow \cos\frac{(2k+2)\pi}{3} + i\sin\frac{(2k+2)\pi}{3} = 1 \\ &\Leftrightarrow k = 3n + 2 \in \{0, ..., 2020\} \Leftrightarrow k \in \{2, 5, 8, ..., 2018\} \end{aligned}$$

Two matrices that verify the given relations are not similar if and only if the numbers k_1, k_2 corresponding to those matrices are different, so the required maximum number of matrices is 673. □

Problem 2. Let $k > 1$ be a real number. Calculate:

(a) $L = \lim_{n\to\infty} \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x.$

(b) $\lim_{n\to\infty} n \left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x \right].$

Proof. **(a)** The limit equals $\boxed{\frac{k}{k-1}}$.

Using the substitution $x = y^n$ we have that

$$I_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x = nk^n \int_0^1 \left(\frac{y}{y + k - 1} \right)^{n-1} \frac{\mathrm{d}y}{y + k - 1}.$$

Using the substitution $\frac{y}{y+k-1} = t \Rightarrow y = \frac{(k-1)t}{1-t}$ we get, after some calculations, that

$$I_n = nk^n \int_0^{\frac{1}{k}} \frac{t^{n-1}}{1-t} \mathrm{d}t.$$

We integrate by parts and we have that

$$I_n = \frac{k}{k-1} - k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2} \mathrm{d}t.$$

It follows that $\lim_{n\to\infty} I_n = \frac{k}{k-1}$ since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2} \mathrm{d}t < \frac{k^{n+2}}{(k-1)^2} \int_0^{\frac{1}{k}} t^n \, \mathrm{d}t = \frac{k}{(n+1)(k-1)^2}.$$

(b) The limit equals $\boxed{\frac{k}{(k-1)^2}}$.

We have that

$$\frac{k}{k-1} - I_n = k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2} \mathrm{d}t.$$

We integrate by parts and we have that

$$\frac{k}{k-1} - I_n = \frac{1}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3} \mathrm{d}t.$$

This implies that

$$\lim_{n\to\infty} n \left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x \right] = \lim_{n\to\infty} \left[\frac{n}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3} \mathrm{d}t \right].$$

Thus

$$\lim_{n\to\infty} n \left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x \right] = \frac{k}{(k-1)^2},$$

since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3} \mathrm{d}t < \frac{k^{n+3}}{(k-1)^3} \int_0^{\frac{1}{k}} t^{n+1} \, \mathrm{d}t = \frac{k}{(k-1)^3(n+2)}.$$

□

Problem 3. Let n be a positive integer, $k \in \mathbb{C}$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{Tr} A \neq 0$ and

$$\operatorname{rank} A + \operatorname{rank}\left((\operatorname{Tr} A) \cdot I_n - kA\right) = n.$$

Find rank A.

Proof. For simplicity, denote $\alpha = \operatorname{Tr} A$. Consider the block matrix:

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \alpha I_n - kA \end{array}\right].$$

We perform on M a sequence of elementary transformations on rows and columns (that do not change the rank) as follows:

$$M \xrightarrow{R_1} \left[\begin{array}{c|c} A & 0 \\ \hline A & \alpha I_n - kA \end{array}\right] \xrightarrow{C_1} \left[\begin{array}{c|c} A & kA \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{R_2}$$
$$\xrightarrow{R_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{C_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline 0 & \alpha I_n \end{array}\right] = N$$

where

R_1 : is the left multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline I_n & I_n \end{array}\right]$;

C_1 : is the right multiplication by $\left[\begin{array}{c|c} I_n & kI_n \\ \hline 0 & I_n \end{array}\right]$;

R_2 : is the left multiplication by $\left[\begin{array}{c|c} I_n & -\frac{k}{\alpha}A \\ \hline 0 & I_n \end{array}\right]$;

C_2 : is the right multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline -\frac{1}{\alpha}A & I_n \end{array}\right]$.

It follows that

$$\operatorname{rank} A + \operatorname{rank}(\alpha I_n - kA) = \operatorname{rank} M = \operatorname{rank} N = \operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) + n.$$

Note that

$$\operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) = 0 \Leftrightarrow A - \frac{k}{\alpha}A^2 = 0 \Leftrightarrow \underbrace{\frac{k}{\alpha}A}_{B} = \left(\frac{k}{\alpha}A\right)^2 \Leftrightarrow B = B^2$$
$$\Rightarrow \operatorname{rank} B = \operatorname{Tr} B = \operatorname{Tr}\left(\frac{k}{\alpha}A\right) = \frac{k}{\alpha}\operatorname{Tr} A = k$$

so finally rank $A =$ rank $B = k$. $\square$

Problem 4. Consider $0 < a < T$, $D = \mathbb{R} \setminus \{kT + a \mid k \in \mathbb{Z}\}$, and let $f : D \to \mathbb{R}$ a $T-$periodic and differentiable function which satisfies $f' > 1$ on $(0, a)$ and

$$f(0) = 0, \ \lim_{\substack{x \to a \\ x < a}} f(x) = +\infty \text{ and } \lim_{\substack{x \to a \\ x < a}} \frac{f'(x)}{f^2(x)} = 1.$$

(a) Prove that for every $n \in \mathbb{N}^*$, the equation $f(x) = x$ has a unique solution in the interval $(nT, nT + a)$, denoted x_n.

(b) Let $y_n = nT + a - x_n$ and $z_n = \displaystyle\int_0^{y_n} f(x)dx$. Prove that $\lim_{n \to \infty} y_n = 0$ and study the convergence of the series $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} z_n$.

Proof. **(1)** Observe first that, for every $n \in \mathbb{N}^*$, $f(nT) = 0$ and $\lim_{\substack{x \to nT+a \\ x < nT+a}} f(x) = +\infty$, hence the equation $f(x) = x$ has at least one solution in the interval $(nT, nT + a)$.

Now, consider the function $g(x) = f(x) - x$ on $(nT, nT + a)$ and observe that if there would exist two solutions of the equation $f(x) = x$, say $x_n^1 < x_n^2$, by Rolle's Theorem, there exists $r_n \in (x_n^1, x_n^2) \subset (nT, nT + a)$ such that $g'(r_n) = f'(r_n) - 1 = 0$, a contradiction, since $f' > 1$ on $(nT, nT + a)$ by periodicity.

(2) Observe that for any n, f is strictly increasing on $(nT, nT + a)$. We prove that (y_n) is decreasing. By contradiction, suppose that $y_n < y_{n+1}$ for some n. Then $T + x_n > x_{n+1}$, and by the monotonicity of f that

$$x_n = f(x_n) = f(x_n + T) > f(x_{n+1}) = x_{n+1},$$

an obvious contradiction.

Since $y_n \in (0, a)$ for every n, it follows that (y_n) it converges. Then there exists $\overline{y} \geq 0$ such that $y_n \to \overline{y}$. Suppose, by contradiction, that $\overline{y} > 0$. Observe that $\overline{y} < a$. Since $x_n - nT \to a - \overline{y}$ for $n \to \infty$, it follows by the continuity of f on $(-T, a)$ that $f(x_n - nT) \to f(a - \overline{y}) \in \mathbb{R}$ for $n \to \infty$. But $f(x_n - nT) = f(x_n) = x_n \to \infty$, hence we obtain a contradiction. Therefore, $y_n \to 0$.

Next, we will prove that

$$\lim_{n \to \infty} n \cdot y_n = \frac{1}{T},$$

hence $\sum_{n=1}^{\infty} y_n$ diverges by a comparison test.

For that, observe that

$$\lim_{n \to \infty} n \cdot y_n = \lim_{n \to \infty} \frac{nT}{Tx_n} \cdot x_n y_n = \frac{1}{T} \lim_{n \to \infty} x_n y_n.$$

Moreover,

$$\begin{aligned} \lim_{n \to \infty} x_n y_n &= \lim_{n \to \infty} f(x_n) \cdot y_n = \lim_{n \to \infty} f(nT + a - y_n) \cdot y_n \\ &= \lim_{n \to \infty} \frac{y_n}{\frac{1}{f(a - y_n)}} = -\lim_{n \to \infty} \frac{(a - y_n) - a}{\frac{1}{f(a - y_n)}}. \end{aligned}$$

But $a - y_n$ converges increasingly to a so the previous limit is

$$-\lim_{\substack{x \to a \\ x < a}} \frac{x - a}{\frac{1}{f(x)}} = -\lim_{\substack{x \to a \\ x < a}} \frac{1}{-\frac{f'(x)}{f^2(x)}} = 1.$$

For the second series, observe that for every n, there is $c_n \in (0, y_n)$ such that $z_n = y_n \cdot f(c_n)$. Since f is increasing on $(0, a)$,

$$z_n \leq y_n \cdot f(y_n) = y_n^2 \cdot \frac{f(y_n)}{y_n}.$$

But f is differentiable at 0, and $\dfrac{f(y_n)}{y_n} \to f'(0) \geq 0$ for $n \to \infty$, hence there exists $M > 0$ such that, for any large n,

$$\frac{f(y_n)}{y_n} \leq M.$$

Then there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$0 \leq z_n \leq \frac{K}{n^2}, \quad \forall n \geq n_0.$$

By a comparison test, $\sum_{n=1}^{\infty} z_n$ converges. □